

THE CONE PERCOLATION MODEL ON SPHERICALLY SYMMETRIC TREES AND ITS VARIATIONS

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ABSTRACT. We study a rumour model from a percolation theory and branching process point of view. The existence of a giant component is related to the event where the rumour, which started from the root of a tree, spreads out through an infinite number of its vertices. We present lower and upper bounds for the probability of that event, according to the distribution of the random variables that defines the radius of influence of each individual. We work with spherically symmetric trees which includes homogeneous and k -periodic trees.

1. INTRODUCTION AND BASIC DEFINITIONS

Since the seminal work of Benjamini and Schram [2], percolation theory beyond the nearest neighbor independent setup on \mathbb{Z}^d has become a very active field of research.

Lebensztayn and Rodriguez [8], introduced a disk percolation model on general graphs where a reaction chain starting from the origin of the graph, based on independent copies of a geometric random variables, may lead to the existence of a giant component.

This line of research was continued by Junior *et al* [6] and [7], focusing on \mathbb{N} and on the homogeneous tree respectively, studying a family of dependent long range (not necessarily homogeneous) percolation model. They studied the criticality of each model, presenting sufficient

Date: October 13, 2015.

2000 Mathematics Subject Classification. 60K35, 60G50.

Key words and phrases. coverage of space, epidemic model, disk-percolation, rumour model.

Research supported by CNPq (310829/2014-3) and FAPESP (09/52379-8).

conditions under which the processes reach a giant component with positive probability. Besides they presented bounds for the probability of having a giant component based on what they considered the radius of influence of each vertex of \mathbb{N} .

Gallo *et al* [4] computed precisely the probability of having a giant component for the homogeneous version of one of the models proposed in Junior *et al* [6], and obtained information about the distribution of the range of the cluster of the origin when it is finite. Besides that, they obtained a law of large numbers and a central limit theorem for the proportion of the cluster of the origin in a range of size n as n diverges. The key step of of proofs presented in Gallo *et al* [4] is to show that, in each model, the vertices belonging to the cluster of the origin can be related to a suitably chosen discrete renewal process. Related results have been obtained recently by Bertachi and Zucca [3] .

Here we focus on homogeneous, periodic and spherically symmetric trees in a process which considers non-negative discrete random variables and their distributions for *radius of influences*. Along the paper we use the letter R to refer to that random variable and to make formulas neater we define $p_k = \mathbb{P}(R = k)$ for $k = 0, 1, \dots$. To avoid trivialities we assume throughout this paper that $p_0 \in (0, 1)$. A graph G is said a *tree* if for any pair of its vertices there is one and only one path (a subset of edges) connecting them. By $|A|$ we denote the cardinality of A . The *degree* of a vertex is the cardinality of its set of neighbors. For two vertices u, v let $d(u, v)$, be the distance between u and v , that is the number of edges the path from u to v has. In order to make some definitions shorter assume that $\sum_{j=0}^n a_j = 0$ for all $n \in \{1, 2, \dots\}$.

Consider a tree \mathbb{T} and its set of vertices $\mathcal{V}(\mathbb{T})$. Single out one vertex from $\mathcal{V}(\mathbb{T})$ and call this \mathcal{O} , the origin of $\mathcal{V}(\mathbb{T})$. For each two vertices $u, v \in \mathcal{V}(\mathbb{T})$, consider that $u \leq v$ if u belongs to the path connecting \mathcal{O} to v .

For a tree \mathbb{T} and $n \geq 1$ we define

$$T^u := \{v \in \mathcal{V} : u \leq v\},$$

$$T_n^u := \{v \in T^u : d(v, \mathcal{O}) \leq d(u, \mathcal{O}) + n\}$$

and

$$M_n(u) := |\partial T_n^u| := |\{v \in T^u : d(v, \mathcal{O}) = d(u, \mathcal{O}) + n\}|.$$

As in Junior *et al* [7], we say that the process *survives* if the amount of vertices involved is infinite. Otherwise we say the process *dies out*. Our main interest is to establish whether the process has positive probability of involving an infinite set of individuals. Besides we present lower and upper bounds for the probability of that event, according to the distribution of the random variables that defines the radius of influence of each individual.

The paper is organized as follows. Sections 2, 3 and 4 present the main results and specific setups and distributions for the Cone Percolation model on Homogeneous Trees, Periodic Trees and Spherically Symmetric Trees respectively. Section 5 brings the proofs for the main results presented along sections 2, 3 and 4 together with auxiliary lemmas and useful inequalities.

2. HOMOGENEOUS TREES

Definition 2.1. We say that a tree, \mathbb{T}_d , is *homogeneous*, if each of its vertices has degree $d + 1$.

Now, let us define

$$\mathbb{T}_d^+(u) = \{v \in \mathcal{V}(\mathbb{T}_d) : u \leq v\}.$$

Next we define the *Cone Percolation Process* in \mathbb{T}_d .

Definition 2.2. The *Cone Percolation Process*.

Let $\{R_v\}_{v \in \mathcal{V}(\mathbb{T}_d)}$ and R be a set of independent and identically distributed random variables. Furthermore, for each $u \in \mathcal{V}(\mathbb{T}_d)$, we define the random sets

$$B_u = \{v \in \mathcal{V}(\mathbb{T}_d) : u \leq v \text{ and } d(u, v) \leq R_u\}. \quad (2.1)$$

With these sets we define the *Cone Percolation Process*, the non-decreasing sequence of random sets $I_0 \subset I_1 \subset \dots$ defined as $I_0 = \{\mathcal{O}\}$ and inductively $I_{n+1} = \bigcup_{u \in I_n} B_u$ for all $n \geq 0$.

Besides, consider $I = \bigcup_{n \geq 0} I_n$ be the connected component of the origin. Under the rumor process interpretation, I is the set of vertices which heard about the rumor. We say that the process *survives* if $|I| = \infty$, referring to the surviving event as V .

Definition 2.3. *Rooted homogeneous tree.*

Pick a $v \in \mathcal{V}(\mathbb{T}_d)$ such that $d(\mathcal{O}, v) = 1$. We define $\mathbb{T}_d^+ = \mathbb{T}_d \setminus \mathbb{T}_d^+(v)$.

Consider \mathbb{P}_+ and \mathbb{P} the probability measures associated to the processes on \mathbb{T}_d^+ and \mathbb{T}_d (we do not mention the random variable R unless

absolutely necessary). By a coupling argument one can see that for a fixed distribution of R

$$\mathbb{P}_+(V) \leq \mathbb{P}(V). \quad (2.2)$$

Furthermore, by the definition of \mathbb{T}_d^+ and its relation with \mathbb{T}_d we have that for a fixed distribution of R

$$\mathbb{P}_+(V) = 0 \text{ if and only if } \mathbb{P}(V) = 0. \quad (2.3)$$

Theorem 2.4. For $dp_0 \leq 1$ and $\mathbb{E}(d^R) < 2 - \frac{1}{d}$, we have

$$\frac{d + \mathbb{E}(d^R) - p_0}{d[1 - \mathbb{E}(d^R) + p_0]} \leq \mathbb{E}(|I|) \leq \frac{\mathbb{E}(d^R) + d - 2}{2d - 1 - d\mathbb{E}(d^R)}.$$

Example 2.5. For $R \sim \text{Bernoulli}(p)$ with $pd < 1$, we have

$$\mathbb{E}(|I|) = \frac{1 + p}{1 - dp}.$$

Example 2.6. For $R \sim \mathcal{G}(p)$ and $pd < \frac{1}{2}$, we have

$$\frac{1 - dp + p - p^2}{1 - 2dp + dp^2} \leq \mathbb{E}(|I|) \leq \frac{1 - dp - p}{1 - 2dp}.$$

That gives us a fairly sharp bound even when we pick p and d such that pd is very close to $\frac{1}{2}$ as, for example, $p = 10^{-6}$ and $d = 499,000$.

For these parameters we get $250.438 \leq \mathbb{E}(|I|) \leq 250.501$.

Example 2.7. For $R \sim B(n, p)$ and $p < \frac{1}{d-1}[\sqrt[n]{\frac{2d-1}{d}} - 1]$, we have

$$\frac{d + (dp + 1 - p)^n - (1 - p)^n}{d[1 - (dp + 1 - p)^n + (1 - p)^n]} \leq \mathbb{E}(|I|) \leq \frac{(dp + 1 - p)^n + d - 2}{2d - 1 - d(dp + 1 - p)^n}.$$

Assuming

$d = 1,000$, $n = 2$ and $p = 4 \times 10^{-4}$ we have $24.825 \leq \mathbb{E}(|I|) \leq 24.924$.

Example 2.8. For $R \sim \mathcal{P}(\lambda)$ and $\lambda < \ln(\sqrt[d-1]{2 - \frac{1}{d}})$, we have

$$\frac{d + e^{(d-1)\lambda} - e^{-\lambda}}{d[1 - e^{(d-1)\lambda} + e^{-\lambda}]} \leq \mathbb{E}(|I|) \leq \frac{e^{(d-1)\lambda} + d - 2}{2d - 1 - de^{(d-1)\lambda}}.$$

In particular, if

$d = 1,000$ and $\lambda = 6 \times 10^{-4}$, we find $5.613 \leq \mathbb{E}(|I|) \leq 5.625$.

3. PERIODIC TREES

Let $\tilde{d} = (d_1, d_2, \dots, d_k)$ be a k -dimensional integer vector such that $d_i \geq 2$ for all $i = 1, 2, \dots, k$.

Definition 3.1. We say that a tree, $\mathbb{T}_{\tilde{d}}$, is *k-periodic*, if any vertex whose shortest path from it to the origin using $nk + i - 1$ edges, for $i = 1, 2, \dots, k$ and $n \in \mathbb{N}$, has degree $d_i + 1$.

A few useful quantities to present the results in this section are

$$d_{(i)} = \text{the } i\text{-th smallest value in } \tilde{d},$$

$$G = G(\tilde{d}) := \sqrt[k]{\prod_{j=1}^k d_j},$$

$$c_0 := 1 \text{ and } c_i := \frac{\prod_{j=1}^i d_{(j)}}{\sqrt[k]{\prod_{j=1}^k (d_j)^i}} = \frac{\prod_{j=1}^i d_{(j)}}{G^i}, i = 1, \dots, k-1;$$

$$\bar{c}_0 := 1 \text{ and } \bar{c}_i := \frac{\prod_{j=k+1-i}^k d_{(j)}}{\sqrt[k]{\prod_{j=1}^k (d_j)^i}} = \frac{\prod_{j=k+1-i}^k d_{(j)}}{G^i}, i = 1, \dots, k-1.$$

Definition 3.2. For $i = 1, \dots, k$ and R , the radius of influence, we define

$$I_i(R) = \begin{cases} 1 & \text{if } R = nk + i \text{ for some } k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Besides, we define

$$\underline{x}_{n,i} := \prod_{j=1}^k (d_j)^n \prod_{j=1}^i d_{(j)} \text{ for } i \neq 0, \underline{x}_{n,0} := \prod_{j=1}^k (d_j)^n \text{ and } \underline{x}_{-1,i} := 0$$

and

$$\bar{x}_{n,i} := \prod_{j=1}^k (d_j)^n \prod_{j=1}^i d_{(k+1-j)} \text{ for } i \neq 0 \text{ and } \bar{x}_{n,0} := \prod_{j=1}^k (d_j)^n.$$

and

$$h_i(R) = \left[\sum_{m=0}^{\lfloor \frac{R-i}{k} \rfloor - 1} \sum_{j=0}^{k-1} (\underline{x}_{m,j})^{-1} + \sum_{j=0}^{i-1} (\underline{x}_{\lfloor \frac{R-i}{k} \rfloor, j})^{-1} \right] G^R.$$

Analogously to definitions 2.2 and 2.3, we consider the *Cone Percolation Process* on $\mathbb{T}_{\tilde{d}}$. Relations analogous to (2.2) and (2.3) also holds between $\mathbb{T}_{\tilde{d}}$ and $\mathbb{T}_{\tilde{d}}^+$.

Theorem 3.3. *Consider the Cone Percolation Process on $\mathbb{T}_{\tilde{d}}^+$ with radius of influence R*

(I) *If*

$$\sum_{i=0}^{k-1} c_i \mathbf{E}(G^R I_i(R)) > 1 + p_0$$

then, $\mathbb{P}_+(V) > 0$,

(II) *If*

$$\sum_{i=0}^{k-1} \bar{c}_i \mathbf{E}(h_i(R) I_i(R)) \leq 1$$

then, $\mathbb{P}_+(V) = 0$.

Corollary 3.4. *Consider the Cone Percolation Process on \mathbb{T}_d^+ (the d -dimensional rooted homogeneous tree) with radius of influence R*

(I) *If $\mathbf{E}(d^R) > 1 + p_0$ then, $\mathbb{P}_+(V) > 0$,*

(II) *If $\mathbf{E}(d^R) \leq 2 - \frac{1}{d}$ then, $\mathbb{P}_+(V) = 0$.*

Let ρ and ψ be, respectively, the smallest non-negative root of the equations

$$\sum_{i=0}^{k-1} \mathbf{E}(\rho^{c_i G^R} I_i(R)) + (1 - \rho)p_0 = \rho, \quad (3.1)$$

$$\sum_{i=0}^{k-1} \mathbf{E}(\psi^{\lfloor \bar{c}_i h(R) \rfloor} I_i(R)) = \psi, \quad (3.2)$$

Theorem 3.5. *Consider the Cone Percolation Process on \mathbb{T}_d^+ . Then,*

$$1 - \rho \leq \mathbb{P}_+(V) \leq 1 - \psi.$$

Theorem 3.6. *For the Cone Percolation Process on $\mathbb{T}_{\tilde{d}}$ with radius of influence R , it holds that*

$$1 - \sum_{i=0}^{k-1} \mathbf{E}(\rho^{M_R(\mathcal{O})} I_i(R)) \leq \mathbb{P}(V) \leq 1 - \sum_{i=0}^{k-1} \mathbf{E}(\psi^{|T_R^\mathcal{O}|} I_i(R)).$$

Corollary 3.7. *For the Cone Percolation Process on \mathbb{T}_d (the d -dimensional homogeneous tree) with radius of influence R , it holds that*

$$1 - \left(1 - \rho^{\frac{d+1}{d}}\right) p_0 - \mathbf{E}\left(\rho^{\frac{(d+1)}{d}d^R}\right) \leq \mathbb{P}(V) \leq 1 - \mathbf{E}\left(\psi^{\frac{(d+1)}{d-1}(d^R-1)}\right)$$

where ρ and ψ are the smallest non-negative root of the equations (3.1) and (3.2).

Example 3.8. Consider a Cone Percolation Process in $\mathbb{T}_{\tilde{d}}$, $\tilde{d} = (4, 9)$ assuming

$$\mathbb{P}(R = k) = (1 - p)p^k, k = 0, 1, 2, \dots$$

In other words $R \sim \mathcal{G}(1 - p)$. From Theorem 3.3 and equation (2.3)

$$0.078542 \leq \inf\{p : \mathbb{P}(V) > 0\} \leq 0.097374.$$

Example 3.9. Consider a Cone Percolation Process in $\mathbb{T}_{\tilde{d}}$, with $\tilde{d} = (12, 15, 16)$. Assuming $R \sim \mathcal{B}(3, 0.1)$, from Theorem 3.6 we have,

$$0.266557 \leq \mathbb{P}(V) \leq 0.266894.$$

4. SPHERICALLY SYMMETRIC TREES

Definition 4.1. We say that a tree, \mathbb{T}_S , is *spherically symmetric*, if any pair of vertices at the same distance from the origin, have the same degree.

Analogously to definitions 2.2 and 2.3, we consider the *Cone Percolation Process* on \mathbb{T}_S .

Definition 4.2. Let us define for a tree \mathbb{T}

$$\dim \inf \partial \mathbb{T} := \lim_{n \rightarrow \infty} \min_{v \in \mathcal{V}} \frac{1}{n} \ln M_n(v).$$

Observe that

$$\dim \inf \partial \mathbb{T}_d = \ln d.$$

Definition 4.3.

$$\rho_n := \prod_{k=0}^{n-1} [1 - \prod_{i=0}^k \mathbb{P}(R < i + 1)].$$

Theorem 4.4. For a Cone Percolation Process in \mathbb{T}_S and R , the radius of influence, $\mathbb{P}(V) > 0$ if

$$\lim_{n \rightarrow \infty} \sqrt[n]{\rho_n} > e^{-\dim \inf \partial \mathbb{T}_S}.$$

Corollary 4.5. For a Cone Percolation Process in \mathbb{T}_S and R , a radius of influence satisfying $\mathbb{P}(R \leq k) = 1$ for some $k \in \mathbb{N}$, $\mathbb{P}(V) > 0$ if

$$\dim \inf \partial \mathbb{T}_S > \ln \left[\frac{1}{1 - \prod_{j=1}^k \mathbb{P}(R < j)} \right].$$

Corollary 4.6. For a Cone Percolation Process in \mathbb{T}_S and R , a radius of influence satisfying

$$\mathbb{P}(R = k) = \frac{Z_\alpha}{(k+1)^\alpha}, \quad k = 1, 2, \dots$$

if $\dim \inf \partial \mathbb{T}_S > 0$, then $\mathbb{P}(V) > 0$.

Example 4.7. Consider a Cone Percolation Process in \mathbb{T}_S and R , a radius of influence satisfying

$$\mathbb{P}(R = 1) = p = 1 - \mathbb{P}(R = 0).$$

In other words $R \sim \mathcal{B}(p)$.

- If $\dim \inf \partial \mathbb{T}_S > -\ln p$ then, $\mathbb{P}(V) > 0$,
- If $\mathbb{T}_S = \mathbb{T}_{\tilde{d}}$ and $G(\tilde{d}) > \frac{1}{p}$ then, $\mathbb{P}(V) > 0$.

5. PROOFS

5.1. Homogeneous Trees.

Proof of Theorem 2.4

Consider the branching processes $\{\mathcal{X}_n\}_{n \in \mathbb{N}}$ and $\{\mathcal{Y}_n\}_{n \in \mathbb{N}}$ defined in [7]. For these processes the average number of offsprings are respectively $\mu_X = \mathbb{E}(d^R) - p_0$ and $\mu_Y = \frac{d}{d-1} [\mathbb{E}(d^R) - 1]$. As $\mu_X < 1$ and $\mu_Y < 1$ by hypothesis, the expected values for the total number of individuals are respectively

$$\frac{1}{1 - \mu_X} = \frac{1}{1 + p_0 - \mathbb{E}(d^R)}, \text{ and}$$

$$\frac{1}{1 - \mu_Y} = \frac{d - 1}{2d - 1 - d\mathbb{E}(d^R)}$$

Consider a small modification in processes $\{\mathcal{X}_n\}_{n \in \mathbb{N}}$ and $\{\mathcal{Y}_n\}_{n \in \mathbb{N}}$ such that the offspring distributions for the first generation are respectively

$$\mathbb{P}[X = 0] = p_0,$$

$$\mathbb{P}[X = (d + 1)d^{k-1}] = p_k \text{ for } k = 1, 2, \dots$$

and

$$\mathbb{P}\left[Y = \frac{(d + 1)(d^k - 1)}{d - 1}\right] = p_k \text{ for } k = 0, 1, 2, \dots$$

For these modified processes the total expected number of individuals are respectively

$$\mathbb{E}(|I_x|) = \sum_{k=1}^{\infty} \left(\frac{(d+1)d^{k-1}}{1+p_0-\mathbb{E}(d^R)} + 1 \right) p_k + p_0 = \frac{d + \mathbb{E}(d^R) - p_0}{d(1 - \mathbb{E}(d^R) + p_0)}$$

and

$$\begin{aligned} \mathbb{E}(|I_y|) &= \sum_{k=0}^{\infty} \left(\left\lceil \frac{(d+1)(d^k-1)}{(d-1)} \right\rceil \left\lceil \frac{(d-1)}{2d-1-d\mathbb{E}(d^R)} \right\rceil + 1 \right) p_k \\ &= \frac{\mathbb{E}(d^R) + d - 2}{2d - 1 - d\mathbb{E}(d^R)}. \end{aligned}$$

Analogously to the reasoning presented in the proof of Theorem 2.3 in [7], we have that $\mathbb{E}(|I_x|) \leq \mathbb{E}(|I|) \leq \mathbb{E}(|I_y|)$ and the result follows. \square

5.2. Periodic Trees.

Consider a k -periodic tree whose degrees are $d_1+1, d_2+1, \dots, d_k+1$ and for $i = 1, \dots, k-1$

$$J_i = \{(j_1, \dots, j_k), 1 \leq j_1 < j_2 < \dots < j_i \leq k\}.$$

Let's define for $n \in \mathbb{N}$

$$\begin{aligned} A_{nk} &= \left\{ \prod_{j=1}^k (d_j)^n \right\}, \\ A_{nk+i} &= \left\{ \prod_{j=1}^k (d_j)^n \prod_{l=1}^i d_{j_l}, (j_1, \dots, j_i) \in J_i \right\} \text{ for } i = 1, \dots, k-1. \end{aligned}$$

We claim that for all $n \in \mathbb{N}$, $k \in \mathbb{N}$ and $v \neq \mathcal{O}$ that

$$\min A_{nk+i} = \underline{x}_{n,i}, \tag{5.1}$$

$$\max A_{nk+i} = \bar{x}_{n,i}, \tag{5.2}$$

$$M_{nk+i}(v) \in A_{nk+i}. \tag{5.3}$$

Let

$$y_{n,i} := \sum_{m=0}^{n-1} \sum_{j=0}^{k-1} (\underline{x}_{m,j})^{-1} + \sum_{j=0}^{i-1} (\underline{x}_{m,j})^{-1}$$

Lemma 5.1. *Consider a k -periodic tree whose degrees are $d_1 + 1, d_2 + 1, \dots, d_k + 1$, $d_i \geq 2$ for all $i = 1, 2, \dots, k$. Consider a vertex $v \neq \mathcal{O}$. Then*

$$|T_{nk+i}^v| \leq \lfloor y_{n,i} \cdot \bar{x}_{n,i} \rfloor.$$

Proof of Lemma 5.1

From (5.2) and (5.3) it follows that

$$\begin{aligned} |T_{nk+i}^v| &\leq \left\lfloor \bar{x}_{n,i} + \cdot + \frac{\bar{x}_{n,i}}{\prod_{j=1}^{k-1} d_{(j)}} + \frac{\bar{x}_{n,i}}{\prod_{j=1}^{k-1} d_{(j)} d_{(1)}} + \cdot + \frac{\bar{x}_{n,i}}{\prod_{j=1}^{k-1} d_{(j)} \prod_{j=1}^{i-1} d_{(j)}} \right\rfloor \\ &= \lfloor y_{n,i} \cdot \bar{x}_{n,i} \rfloor. \end{aligned}$$

□

Let us define two auxiliary branching process, being the first one $\{\mathcal{X}_n\}_{n \in \mathbb{N}}$. This process is defined by a random variable X , assuming values in $\{\underline{x}_{n,i}, i = 0, \dots, k-1, \text{ and } n = 0, 1, \dots, (n, i) \neq (0, 0)\} \cup \{0\}$ such that

$$\mathbf{P}[X = 0] =: p_0,$$

$$\mathbf{P}[X = \underline{x}_{n,i}] =: p_{nk+i} \text{ for } i = 0, \dots, k-1, \text{ and } n = 0, 1, \dots, (n, i) \neq (0, 0)$$

Its expected value is given by the following lemma

Lemma 5.2.

$$\mathbf{E}[X] = \sum_{i=0}^{k-1} c_i \mathbb{E}[G^R I_i(R)] - p_0$$

Proof of Lemma 5.2

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{n=1}^{\infty} x_{n,0} p_{nk} + \sum_{i=1}^{k-1} \sum_{n=0}^{\infty} x_{n,i} p_{nk+i} \\
&= \sum_{i=0}^{k-1} c_i \sum_{n=0}^{\infty} \prod_{j=1}^k (\sqrt[k]{d_j})^{nk+i} p_{nk+i} - p_0 \\
&= \mathbb{E}[G^R I_0(R)] + \sum_{i=1}^{k-1} c_i \mathbb{E}[G^R I_i(R)] - p_0 \\
&= \sum_{i=0}^{k-1} c_i \mathbb{E}[G^R I_i(R)] - p_0.
\end{aligned}$$

□

and its probability generating function is given by

Lemma 5.3.

$$\varphi_X(s) = \sum_{i=0}^{k-1} \mathbb{E} \left[s^{c_i G^R} I_i(R) \right] + (1-s)p_0.$$

Proof of Lemma 5.3

$$\begin{aligned}
\varphi_X(s) &= p_0 + \sum_{n=1}^{\infty} s^{x_{n,0}} p_{nk} + \sum_{i=1}^{k-1} \sum_{n=0}^{\infty} s^{x_{n,i}} p_{nk+i} \\
&= p_0 + \sum_{n=1}^{\infty} s^{G^{nk}} p_{nk} + \sum_{i=1}^{k-1} \sum_{n=0}^{\infty} s^{c_i G^{nk+i}} p_{nk+i} \\
&= p_0 - s p_0 + \sum_{i=0}^{k-1} \sum_{n=0}^{\infty} s^{c_i G^{nk+i}} p_{nk+i} \\
&= (1-s)p_0 + \sum_{i=0}^{k-1} \mathbb{E} \left[s^{c_i G^R} I_i(R) \right].
\end{aligned}$$

The second auxiliary process is $\{\mathcal{Y}_n\}_{n \in \mathbb{N}}$, a branching process defined by a random variable Y , assuming values on $\{\lfloor y_{n,i} \bar{x}_{n,i} \rfloor, i = 0, \dots, k -$

1, and $n = 0, 1, \dots\}$ such that

$$\mathbf{P}\left[Y = \lfloor y_{n,i} \bar{x}_{n,i} \rfloor\right] = p_{nk+i} \text{ for } i = 0, 1, \dots, k-1 \text{ and } n = 0, 1, \dots$$

Its expected value satisfies

Lemma 5.4.

$$\mathbf{E}[Y] \leq \sum_{i=0}^{k-1} \bar{c}_i \mathbf{E}[h_i(R) I_i(R)].$$

Proof of Lemma 5.4

$$\begin{aligned} \mathbf{E}(Y) &\leq \sum_{i=0}^{k-1} \sum_{n=0}^{\infty} y_{n,i} \bar{x}_{n,i} p_{nk+i} \\ &= \sum_{i=0}^{k-1} \sum_{n=0}^{\infty} \left[\sum_{m=0}^{n-1} \sum_{j=0}^{k-1} (\underline{x}_{m,j})^{-1} + \sum_{j=0}^{i-1} (\underline{x}_{m,j})^{-1} \right] \bar{x}_{n,i} p_{nk+i} \\ &= \sum_{i=0}^{k-1} \bar{c}_i \sum_{n=0}^{\infty} \left[\sum_{m=0}^{n-1} \sum_{j=0}^{k-1} (\underline{x}_{m,j})^{-1} + \sum_{j=0}^{i-1} (\underline{x}_{m,j})^{-1} \right] \prod_{j=1}^k (\sqrt[k]{d_j})^{nk+i} p_{nk+i} \\ &= \sum_{i=0}^{k-1} \bar{c}_i \mathbf{E}[h_i(R) I_i(R)]. \end{aligned}$$

□

and its probability generating function is given by

Lemma 5.5.

$$\varphi_Y(s) = \sum_{i=0}^{k-1} \mathbf{E}\left[s^{\lfloor \bar{c}_i h_i(R) \rfloor} I_i(R)\right]. \quad (5.4)$$

Proof of Lemma 5.5

$$\begin{aligned} \varphi_Y(s) &= \sum_{i=0}^{k-1} \sum_{n=0}^{\infty} s^{\lfloor y_{n,i} \bar{x}_{n,i} \rfloor} p_{nk+i} \\ &= \sum_{i=0}^{k-1} \sum_{n=0}^{\infty} s^{\lfloor y_{n,i} G^{nk+i} \bar{c}_i \rfloor} p_{nk+i} \\ &= \sum_{i=0}^{k-1} \mathbf{E}\left[s^{\lfloor \bar{c}_i h_i(R) \rfloor} I_i(R)\right]. \end{aligned}$$

□

Proof of Theorem 3.3

By a coupling argument one can see that our process dominates (by (5.1) and (5.3)) $\{\mathcal{X}_n\}_{n \in \mathbb{N}}$. This process survives as long as $\mathbf{E}[X] > 1$. Therefore from Lemma 5.2 our process survives if

$$\sum_{i=0}^{k-1} c_i \mathbf{E} [G^R I_i(R)] > 1 + \mathbb{P}(R = 0),$$

proving (I).

By the other side, also by a coupling argument, our process is dominated (by (5.2) and (5.3)) by $\{\mathcal{Y}_n\}_{n \in \mathbb{N}}$. That process dies out provided $\mathbf{E}[Y] \leq 1$ therefore from Lemma 5.4 our process dies out if

$$\sum_{i=0}^{k-1} \bar{c}_i \mathbf{E} (h_i(R) I_i(R)) \leq 1,$$

proving (II). □

Proof of Theorem 3.5

In order to find the extinction probability of $\{\mathcal{X}_n\}_{n \in \mathbb{N}}$ (Grimmett and Stirzaker [5, p.173]), let us consider the smallest non-negative root of the equation $\rho = \varphi_X(\rho)$. Therefore from Lemma 5.3

$$\sum_{i=0}^{k-1} \mathbf{E} \left[\rho^{c_i G^R} I_i(R) \right] + (1 - \rho) p_0 = \rho$$

and by construction of the processes, as $\mathbb{P}_+[V^c] \leq \rho$, we have that

$$1 - \rho \leq \mathbb{P}_+(V).$$

In order to find the extinction probability of $\{\mathcal{Y}_n\}_{n \in \mathbb{N}}$ (Grimmett and Stirzaker [5, p.173]), let us consider the smallest non-negative root of the equation $\psi = \varphi_Y(\psi)$. Therefore from Lemma 5.5

$$\sum_{i=0}^{k-1} \mathbf{E} \left[\psi^{\lfloor \bar{c}_i h_i(R) \rfloor} I_i(R) \right] = \psi$$

and by the construction of the processes, as $\mathbb{P}_+[V^c] \geq \psi$, we have that

$$\mathbb{P}_+(V) \leq 1 - \psi.$$

□

Proof of Theorem 3.6

Observe that except for the root, all vertices see towards infinity a tree like \mathbb{T}_d^+ . So, assuming $\bar{R}_\mathcal{O} = nk+i$ the probability for the process to survive is greater or equal than the probability of the process to survive from at least one of more than $M_{nk+i}(\mathcal{O})$ trees that have as root the furthest infected vertices. Now note that, still assuming $\bar{R}_\mathcal{O} = nk+i$, the probability for the process to survive on $\mathbb{T}_{\bar{d}}$ is smaller or equal than the probability for the process to survive from at least one of less than $|T_{nk+i}^\mathcal{O}|$ vertices which are in the radius of influence ($\bar{R}_\mathcal{O}$) of the origin of the tree. Then

$$\mathbb{P}(V|\bar{R}_\mathcal{O} = nk+i) \geq 1 - (1 - \mathbb{P}_+(V))^{M_{nk+i}(\mathcal{O})} \geq 1 - \rho^{M_{nk+i}(\mathcal{O})}$$

and

$$\mathbb{P}(V|\bar{R}_\mathcal{O} = nk+i) \leq 1 - (1 - \mathbb{P}_+(V))^{|T_{nk+i}^\mathcal{O}|} \geq 1 - \psi^{|T_{nk+i}^\mathcal{O}|}.$$

Then,

$$\begin{aligned} \mathbb{P}(V) &= \sum_{i=0}^{k-1} \sum_{n=0}^{\infty} \mathbb{P}(V|R_\mathcal{O} = nk+i)p_{nk+i} \\ &\geq \sum_{i=0}^{k-1} \sum_{n=0}^{\infty} [1 - \rho^{M_{nk+i}(\mathcal{O})}]p_{nk+i} \\ &= 1 - \sum_{i=0}^{k-1} \mathbb{E}[\rho^{M_R(\mathcal{O})} I_i(R)] \end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}(V) &= \sum_{i=0}^{k-1} \sum_{n=0}^{\infty} \mathbb{P}(V | R_{\mathcal{O}} = nk + i) \mathbb{P}(R_{\mathcal{O}} = nk + i) \\
&\leq \sum_{i=0}^{k-1} \sum_{n=0}^{\infty} \left[1 - \psi^{|T_{nk+i}^{\mathcal{O}}|} \right] p_{nk+i} \\
&= 1 - \sum_{i=0}^{k-1} \mathbb{E} \left(\psi^{|T_R^{\mathcal{O}}|} I_i(R) \right).
\end{aligned}$$

□

5.3. Spherically Symmetric Trees.

Suppose we have a set of independent random variables $\{R_v\}_{\{v \in \mathcal{V}(\mathbb{T}_S)\}}$ distributed as R . Assume $\mathbb{P}(R = 0) < 1$.

For $u \leq v \in \mathcal{V}(\mathbb{T}_S)$, consider the event

$$V_{u,v} : \text{Process starting from } u \text{ reaches } v.$$

For a fixed integer n , let $X_0^n = \{\mathcal{O}\}$. Besides, for $j = 1, 2, \dots$ consider

$$X_j^n = \bigcup_{u \in X_{j-1}^n} \{v \in \partial T_n^u : V_{u,v} \text{ occurs}\}.$$

Again, for all $j = 1, 2, \dots$ consider

$$Z_j^n = |X_j^n|.$$

So, for all fixed positive integer n , $\{Z_j^n\}_{j \geq 0}$ is a branching process dominated by the number of vertices $v \in \partial T_{jn}^{\mathcal{O}}$ which are activated.

Lemma 5.6. *Consider n fixed. For μ_j , the mean number of offspring of one individual of generation j for the process $\{Z_j^n\}_{j \geq 0}$, it holds that*

$$\mu_j := \mu_j^n = M_n(u) \rho_j^{(n)},$$

where $\rho_j^{(n)} = \mathbb{P}(V_{u,v})$, for any fixed pair $u \leq v$ such that $d(\mathcal{O}, u) = jn$ and $d(\mathcal{O}, v) = (j+1)n$.

Proof of Lemma 5.6

For fixed j and n , consider for some u such that $d(\mathcal{O}, u) = jn$, $\partial T_n^u = \{v_1, v_2, \dots, v_{M_n(u)}\}$. So we can write the number of offspring of u as $\sum_{i=1}^{M_n(u)} I_{\{V_{u,v_i}\}}$. Taking expectation finishes the proof. \square

Lemma 5.7. *Consider n fixed and $\rho_j^{(n)} = \mathbb{P}(V_{u,v}) = \mathbb{P}(V_n)$, for any fixed pair $u \leq v$ such that $d(\mathcal{O}, u) = jn$ and $d(\mathcal{O}, v) = (j+1)n$,*

$$\rho_j^{(n)} \geq \prod_{k=0}^{n-1} [1 - \prod_{i=0}^k \mathbb{P}(R < i+1)].$$

Proof of Lemma 5.7

For any fixed pair $u \leq v$ such that $d(\mathcal{O}, u) = jn$ and $d(\mathcal{O}, v) = (j+1)n$ we have that

$$V_{u,v} = \bigcap_{k=0}^{n-1} \left[\bigcup_{i=0}^k \{R_{u(i)} \geq k+1-i\} \right]$$

where $u(i)$ is the vertex from the path connecting u to v such that $d(\mathcal{O}, u(i)) = jn + i$. From this follows

$$\begin{aligned} \rho_j^{(n)} &= \mathbb{P} \left(\bigcap_{k=0}^{n-1} \left[\bigcup_{i=0}^k \{R_{u(i)} \geq k+1-i\} \right] \right) \\ &\geq \prod_{k=0}^{n-1} \mathbb{P} \left(\bigcup_{i=0}^k \{R_{u(i)} \geq k+1-i\} \right). \end{aligned}$$

The inequality is a consequence of the FKG inequality (N.Alon and J.Spencer [1, p.89]). \square

Proof of Theorem 4.4

Assume that $\dim \inf \partial \mathbb{T}_S > 0$. Then, for all $\alpha \in (0, \dim \inf \partial \mathbb{T}_S)$ there exists $N = N(\alpha)$ such that for all $n \geq N$

$$\min_{v \in \mathcal{V}} \frac{1}{n} \ln M_n(v) > \alpha$$

donde

$$M_n(v) \geq e^{\alpha n} \text{ para todo } v \in \mathcal{V} \text{ e } n \geq N.$$

From Souza & Biggins ([9, p.40]) a branching process in varying environments is *uniformly supercritical* if there exists constants $a > 0$ and $c > 1$ such that

$$\prod_{k=i}^{j+i-1} \mu_k \geq ac^j, \text{ for all } i \geq 0 \text{ and } j \geq 0.$$

Observe that that condition holds if

$$\liminf_{j \rightarrow \infty} \mu_j > 1$$

From Lemma 5.6 we have that for $n \geq N$

$$\liminf_{j \rightarrow \infty} \mu_j \geq e^{\alpha n} \rho_n = (e^\alpha \sqrt[n]{\rho_n})^n$$

Now note that we can write

$$Z_{j+1} = \sum_{i=1}^{Z_j} Y_{j,i}^n,$$

where $Y_{j,i}^n$ are i.i.d. copies of Y_j^n , being the number of offspring from the i -th individual of the j -th generation. By considering Lemma 5.6 we have for all j that

$$\frac{Y_j^n}{\mu_j} \leq \frac{M_n(u)}{\mu_j} = \frac{1}{\rho_j^{(n)}} \leq (\mathbb{P}[R > 0])^{-n}$$

where $\rho_j^{(n)} = \mathbb{P}(V_{u,v})$, for any fixed pair $u \leq v$ such that $d(\mathcal{O}, u) = jn$ and $d(\mathcal{O}, v) = (j+1)n$.

So, from Theorem 1 in Souza & Biggins ([9, p.40]), we conclude that the cone percolation process has a giant component with positive probability if

$$\lim_{n \rightarrow \infty} \sqrt[n]{\rho_n} > e^{-\alpha}.$$

As this hold for every $\alpha \in (0, \dim \inf \partial \mathbb{T}_S)$, the condition

$$\lim_{n \rightarrow \infty} \sqrt[n]{\rho_n} > e^{-\dim \inf \partial \mathbb{T}_S}$$

guarantees the survival of the process with positive probability. \square

Proof of Corollary 4.5

$$\begin{aligned} & \sqrt[n]{\prod_{i=0}^{n-1} [1 - \prod_{j=0}^i \mathbb{P}(R < j+1)]} = \\ & = [1 - \prod_{j=1}^k \mathbb{P}(R < j)] \sqrt[n]{\frac{\prod_{i=0}^{k-1} [1 - \prod_{j=0}^i \mathbb{P}(R < j+1)]}{(1 - \prod_{j=1}^k \mathbb{P}(R < j))^k}} \\ & \rightarrow 1 - \prod_{j=1}^k \mathbb{P}(R < j), \text{ when } n \rightarrow \infty. \end{aligned}$$

Proof of Corollary 4.6

Observe that

$$\begin{aligned} \rho_n & \geq \mathbb{P}(R \geq n) = \sum_{k=n}^{\infty} \frac{Z_{\alpha}}{(k+1)^{\alpha}} \\ & \geq \int_{n+1}^{\infty} \frac{Z_{\alpha}}{x^{\alpha}} dx = \frac{Z_{\alpha}}{(\alpha-1)(n+1)^{\alpha-1}} \end{aligned}$$

The above inequality follows from the integral test.

Now observe that if $\dim \inf \partial \mathbb{T}_S > 0$, we have that

$$\lim_{n \rightarrow \infty} \sqrt[n]{\rho_n} \geq \lim_{n \rightarrow \infty} \sqrt[n]{\frac{Z_{\alpha}}{(\alpha-1)(n+1)^{\alpha-1}}} = 1 > e^{-\dim \inf \partial \mathbb{T}_S}$$

Theorem 4.4 guarantess the desired result. \square

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